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MAXIMUM CRITICAL MACH NUMBER FLOWS AROUND SEMI-INFINITE SOLIDS OF REVOLUTION†

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The problem of constructing the leading part of a semi-infinite solid of revolution with an axially symmetric flow of an ideal (inviscid and non-heat-conducting) gas around it which has the maximum critical Mach number is considered. A numerical-analytic method is proposed for solving this problem. Results are presented for the case of a perfect gas with an adiabatic index $\gamma = 1.4$.

Among all the bodies which satisfy certain geometric constraints, bodies which achieve the maximum possible critical Mach number do not experience wave drag in the maximum range of free-stream velocities. Below, we shall refer to such bodies as optimal bodies.

The structure of plane symmetric optimal bodies and optimal solids of revolution in a flow of an ideal gas was investigated for the first time in [1] and the investigations were subsequently extended in [2]. It was established that, for an extensive class of geometric constraints, the generatrices of optimal bodies consist of rectilinear segments and segments in which the gas velocity is constant and equal to the critical velocity. In particular, the problem (we shall call it Problem A) of constructing the optimal solid of revolution with a specified thickness-to-length ratio was formulated and it was shown that Problem A reduces to the problem of the axially symmetric gas flow around a disc at the critical velocity on the free surface in accordance with the Ryabushinskii scheme [1]. An efficient numerical-analytic method of calculating the ideal gas flow past a cone with an arbitrary apex angle in accordance with the Ryabushinskii scheme has been proposed in [4]. A solution of Problem A for an ideal gas with an adiabatic index $\gamma = 1.4$ has been obtained in [5].

1. FORMULATION OF THE PROBLEM

We consider the following problem (which we shall call Problem B). Suppose that an ideal gas flows around a circular right cylinder and that the flow is directed along its axis. It is required to deform the leading part of the cylinder (the part adjacent to the front end) so that there is a continuous flow around the resulting solid of revolution at the maximum possible value of the critical Mach number M_* when one of the conditions is satisfied

$$L / R \leqslant l_0, \quad S' / R^2 \leqslant m_0, \quad W' / R^3 \leqslant n_0 \tag{1.1}$$

Here R is the radius of the cylinder, L is the length of the leading part of the body (the deformed apart of the cylinder), S' and W' are the area in the meridian half-plane and the volume which are lost during the deformation of the cylinder and l_0, m_0, n_0 are given constants. Hence $S' = LR - S$, $W' = \pi LR^2 - W$, where S and W are the area in the meridian half-plane and the volume of the leading part of the body. The inequalities (1.1) limit the loss of "imbeddability" of the body which originally existed.

Using the comparison theorem [1, 2], it can be shown that the leading part of an optimal semi-infinite solid of revolution which satisfies one of conditions (1.1) is formed by a disc and the stream surface which joins it to the cylinder, at each point of which the gas velocity is equal to the critical velocity. Problem B therefore reduces to the problem of the axially symmetric gas flow around a disc at the critical velocity on the free surface according to the Zhukovskii–Roshko scheme [3].

Note that the results of calculations of the critical Mach number M_* for certain shapes of the leading part of a semi-infinite solid of revolution can be found in [6–9]. The results of the solution of a problem on the profiling of the leading and trailing sections of a plane symmetric semi-infinite body which provides the maximum value of M_* are presented in [10].

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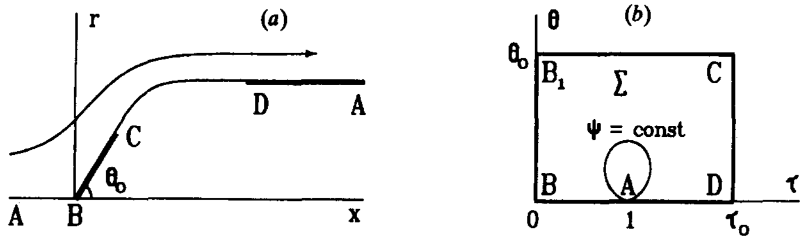

Fig. 1.

2. CALCULATION OF THE SUBSONIC FLOW AROUND A CONE USING THE ZHUKOVSKII-ROSHKO SCHEME

We consider the axially symmetric subsonic flow of a compressible fluid around a circular cone using the Zhukovskii–Roshko scheme. The fluid is assumed to be ideal and weightless and the flow is assumed to be a steady, irrotational and isentropic. The numerical-analytic method for calculating such a flow, which is proposed below, is similar to the method suggested in [4] for investigating the flow around a cone using the Ryabushinskii scheme

In the half-plane of the cylindrical coordinates x, r, the flow domain is bounded by the segment AB of the x axis, the generatrix of the cone BC, which makes an angle θ_0 with the x axis, the arc CD of the free surface and the half-line DA parallel to the x axis (Fig. 1a).

Suppose that λ is the reduced velocity, θ is the angle of inclination of the velocity to the x axis, and λ_a and λ_c are the values of λ at the infinitely distant point A and in CD, respectively, ($\lambda_a < \lambda_c \leqslant 1$), $\tau = \lambda/\lambda_a$, $\tau_0 = \lambda_c/\lambda_a$. In the plane of the variables (τ, θ), the rectangle $\Sigma = \{(\tau, \theta) | 0 < \tau < \tau_0, 0 < \theta < \theta_0\}$ corresponds to the flow domain (Fig. 1b; the segment BB_1 corresponds to the stagnation point B).

Let ρ be the density of the fluid, ρ_0 the value of ρ at the stagnation point, $\nu = \rho/\rho_0$, and M be the Mach number. Moreover, ν and M are known analytic functions of τ, $y = r^2/2$, $Y = \nu\tau y$, and ψ is a stream function which is introduced using the relations

$$\tau\cos\theta = (r\nu)^{-1}\psi_r, \quad \tau\sin\theta = -(r\nu)^{-1}\psi_x$$

(subscripts are used to denote partial derivatives with respect to x, r, τ and θ).

We know [11, 4] that the functions $\psi = \psi(\tau, \theta)$ and $y = y(\tau, \theta)$ satisfy the relations

$$R = R(\psi, Y) = \sin\theta Q^2 L - P_\theta Q + PQ_\theta = 0$$

$$L = L(\psi) = (1-M^2)\psi_{\theta\theta} + \tau^2\psi_{\tau\tau} + (1+M^2)\tau\psi_\tau \tag{2.1}$$

$$P = P(\psi) = \sin^2\theta[\tau^2\psi_\tau^2 + (1-M^2)\psi_\theta^2], \quad Q = Q(\psi, Y) = 2Y + \psi_\theta\sin\theta$$

$$y_\tau = [(M^2-1)\psi_\theta\sin\theta + P/Q + \tau\psi_\tau\cos\theta](\nu\tau^2)^{-1}$$

$$y_\theta = (\tau\psi_\tau\sin\theta + \psi_\theta\cos\theta)(\nu\tau)^{-1} \tag{2.2}$$

The conditions

$$\psi = 0 \text{ in } ABB_1CDA, y = 0 \text{ in } ABB_1, y = y_d \text{ in } AD \tag{2.3}$$

where y_d is a certain positive constant, must be satisfied on the boundary of the domain Σ.

Using (2.2) and (2.3), Y can be expressed in terms of ψ

$$Y = Y(\psi) = \psi\cos\theta + \int_0^\theta (\tau\psi_\tau + \psi)\sin\theta d\theta + \nu\tau\Omega(\tau) \tag{2.4}$$

$$\Omega(\tau) = 0, \quad \tau < 1; \quad \Omega(\tau) = y_d, \quad \tau > 1$$

The problem therefore reduces to determining a function $\psi(\tau, \theta)$ which satisfies relations (2.1) and

(2.4) in the domain Σ, the boundary condition (2.3) and the condition $\psi > 0$ when $(\tau, \theta) \in \Sigma$.

The function ψ is represented in the form $\psi = \psi^0 + \chi$, where ψ^0 is the singular part of the stream function which describes its behaviour in the neighbourhood of the image of the infinitely distant point A. We will seek ψ^0 in the form of an asymptotic expansion in the small parameter θ by putting

$$\psi^0 = \psi_1 + \psi_2 + \psi_3 + \ldots, \quad \psi_k = h_k(\theta) f_k(\omega) \tag{2.5}$$

$$h_{k+1}(\theta) / h_k(\theta) \to 0 \quad \text{for} \quad \theta \to 0, \quad k = 1, 2, \ldots$$

$$\omega = \operatorname{arctg}(\theta / q), \quad q = \alpha\zeta, \quad \zeta = \tau - 1, \quad \alpha = \sqrt{1 - M_a^2} \tag{2.6}$$

and requiring that the following conditions are satisfied

$$\psi_k = 0 \quad \text{на} \quad BAD, \quad k = 1, 2, \ldots; \quad \psi_1 > 0, \quad (\tau, \theta) \in \Sigma \tag{2.7}$$

Here, M_a is the value of M when $\tau = 1$ (the Mach number of the free stream) and $\omega \in (0, \pi)$ when $(\tau, \theta) \in \Sigma$.

The leading term of the asymptotic expansion (2.5) is sought in the form $\psi_1 = \theta^{-n} f_1(\omega)$ (n = const, $n > 0$).

Using (2.6), any analytic function of τ can be represented in the form of a power series in θ with coefficients which depend on ω. In particular

$$\tau = 1 + \alpha^{-1}\theta \operatorname{ctg}\omega, \quad 1 - M^2 = a_0 + a_1\alpha^{-1}\theta \operatorname{ctg}\omega + O(\theta^2)$$

$$\tau(1 + M^2) = c_0 + c_1\alpha^{-1}\theta \operatorname{ctg}\omega + O(\theta^2) \tag{2.8}$$

$$a_0 = \alpha^2, \quad a_1 = -(dM^2 / d\tau)\big|_{\tau=1}, \quad c_0 = 2 - \alpha^2, \quad c_1 = c_0 - a_1$$

Using (2.8) and the relations $\omega_\theta = \theta^{-1} \sin\omega \cos\omega$, $\omega_\tau = -\alpha\theta^{-1} \sin^2\omega$, it can be shown that, when $\psi_1 = \theta^{-n} f_1(\omega)$

$$Y(\psi_1) = \theta^{-n} f_1 + O(\theta^{-n+1}), \quad L(\psi_1) = L_1 + \Delta L_1, \quad P(\psi_1) = P_1 + \Delta P_1$$

$$Q(\psi_1) = Q_1 + \Delta Q_1, \quad R(\psi_1) = R_1 + \Delta R_1$$

$$L_1 = \alpha^2\theta^{-n-2}[(n^2 + n) f_1 - 2nstf_1' + s^2 f_1'']$$

$$P_1 = \alpha^2\theta^{-2n}(n^2 f_1^2 - 2nstf_1 f_1' + s^2 f_1'^2), \quad Q_1 = \theta^{-n}[(2 - n) f_1 + stf_1']$$

$$P_{1\theta} = \alpha^2\theta^{-2n-1}(-2n^3 f_1^2 + st(6n^2 - 2n + 4ns^2) f_1 f_1' +$$

$$+ s^2(2t^2 - 2nt^2 - 2n) f_1'^2 - 2ns^2t^2 f_1 f_1'' + 2s^3 tf_1' f_1'']$$

$$Q_{1\theta} = \theta^{-n-1}[(n^2 - 2n) f_1 + st(3 - 2n - 2s^2) f_1' + s^2t^2 f_1'']$$

$$\Delta L_1 = O(\theta^{-n-1}), \quad \Delta P_1 = O(\theta^{-2n+1}), \quad \Delta Q_1 = O(\theta^{-n+1})$$

$$R_1 = \theta Q_1^2 L_1 - P_{1\theta} Q_1 + P_1 Q_{1\theta} = O(\theta^{-3n-1}), \quad \Delta R_1 = O(\theta^{-3n})$$

where $s = \sin\omega$, $t = \cos\omega$, $Q(\psi) = Q(\psi, Y(\psi))$, $R(\psi) = R(\psi, Y(\psi))$.

On equating R_1, the leading term of the expansion of $R(\psi_1)$ in powers of θ, to zero, we obtain a differential equation in $f_1(\omega)$ from which, after making the substitution $f_1(\omega) = \sin^n\omega\varphi(\omega)$, we arrive at the equation

$$[4 + (n^2 - 4n)s^2]\varphi^2\varphi'' + [(4 + 2n - n^2)s^2 - 4]\varphi\varphi'^2 +$$

$$+ st(\varphi'^3 + n^2\varphi^2\varphi') + (4n^2 - 2n^3)s^2\varphi^3 = 0 \tag{2.9}$$

From (2.7) we obtain the following conditions for $\varphi(\omega)$

$$\varphi(0)=\varphi(\pi)=0; \quad \varphi(\omega)>0, \quad 0<\omega<\pi \tag{2.10}$$

Analysis shows that a unique solution of problem (2.9), (2.10) (apart from a constant factor for φ) exists: $n = 1$, $\varphi = \sin^2\omega$. Hence

$$\psi_1=\theta^{-1}\sin^3\omega=\theta^2(\theta^2+q^2)^{-3/2} \tag{2.11}$$

In the plane of the variables q and θ, the curves $\psi_1 = c = \text{const}$ have the parametric representation $q = c^{-1}\sin^2\omega\cos\omega$, $\theta = c^{-1}\sin^3\omega$. They are similar closed curves with curvature of constant sign which touch the q axis at the origin of coordinates and are symmetrical about the θ axis.

It can be shown that, when $\psi_1 = \theta^{-1}\sin^3\omega$

$$\int\limits_0^\theta \psi_{1\tau}\theta d\theta=\alpha(3\cos\omega-\cos^3\omega-2p), \quad p=\operatorname{sign}(\tau-1)$$

Hence, by (2.4)

$$Y(\psi_1)=\theta^{-1}\sin^3\omega+\alpha(3\cos\omega-\cos^3\omega-2p)+\nu_a\Omega(\tau)+O(\theta)$$

where ν_a is the value of ν when $\tau = 1$. In order that the function $Y(\tau, \theta)$ should preserve its continuity on crossing the line $\tau = 1$, it is necessary to put $\Omega(1) = 2\alpha\nu_a^{-1}$, $\Omega(\tau) = y_d = 4\alpha\nu_a^{-1}$ when $\tau > 1$. Hence

$$Y(\psi_1)=\theta^{-1}\sin^3\omega+\alpha(3\cos\omega-\cos^3\omega+2)+O(\theta)$$

The function $h_1(\theta)$ in the representation $\psi_1 = h_1(\theta)f_1(\omega)$ could have been expressed in more general form than $h_1(\theta) = \theta^{-n}$. By putting $h_1(\theta) = \theta^{-n}(\ln\theta)^m$, for example, and employing the methods used above, $\psi_1 = \theta^{-1}(\ln\theta)^m \sin^3\omega$ would be obtained for arbitrary m = const. The correctness of the result $\psi_1 = \theta^{-1}\sin^3\omega$ is therefore not obvious. In order to confirm its correctness it is necessary to show that it is possible to find the terms in the asymptotic expansion (2.5) following ψ_1.

Substituting $\psi = \psi_1 = \theta^{-1}\sin^3\omega$ into Eq. (2.1), terms of the order of θ^{-4} which arise here cancel out and a residual of the order of θ^{-3} remains. It would be natural to seek the term following ψ_1 in expansion (2.5) in the form $\psi_2 = f_2(\omega)$, requiring that, as a result of substituting $\psi = \psi_1 + \psi_2$ into (2.1), terms of the order of θ^{-3} should be eliminated and a residual of the order of θ^{-2} should be obtained. However, analysis shows that, by proceeding along this path, it is impossible to obtain a function ψ_2 which satisfies condition (2.7). This suggests that, between the terms of the order of θ^{-1} and 1 in expansion (2.5), there is a term of an intermediate order which generates the additional terms of the order of θ^{-3} in the expression $R(\psi^0)$.

Let $\psi_1 = \theta^{-1}\sin^3\omega$. $\psi_2 = \ln\theta f_2(\omega)$, $\psi_0 = \psi_1 + \psi_2$. Then

$$Y(\psi_0)=\theta^{-1}s^3+\ln\theta f_2+O(1), \quad L(\psi_0)=L_1+L_2+\Delta L_2$$

$$P(\psi_0)=P_1+P_2+\Delta P_2, \quad Q(\psi_0)=Q_1+Q_2+\Delta Q_2, \quad R(\psi_0)=R_2+\Delta R_2$$

$$L_1=\alpha^2\theta^{-3}(2s^3-3s^5), \quad P_1=\alpha^2\theta^{-2}(4s^6-3s^8), \quad Q_1=\theta^{-1}(4s^3-3s^5)$$

$$P_{1\theta}=\alpha^2\theta^{-3}(16s^6-42s^8+24s^{10}), \quad Q_{1\theta}=\theta^{-2}(8s^3-24s^5+15s^7)$$

$$L_2=\alpha^2\theta^{-2}\ln\theta s^2 f_2'', \quad P_2=4\alpha^2\theta^{-1}\ln\theta s^4 tf_2', \quad Q_2=\ln\theta(2f_2+stf_2')$$

$$P_{2\theta}=P_{2\theta}^{(1)}+P_{2\theta}^{(2)}, \quad Q_{2\theta}=Q_{2\theta}^{(1)}+Q_{2\theta}^{(2)}$$

$$P_{2\theta}^{(1)}=4\alpha^2\theta^{-2}\ln\theta[s^5t^2f_2''+s^4t(3-5s^2)f_2'], \quad P_{2\theta}^{(2)}=4\alpha^2\theta^{-2}s^4tf_2'$$

$$Q_{2\theta}^{(1)}=\theta^{-1}\ln\theta[st(3-2s^2)f_2'+s^2t^2f_2''], \quad Q_{2\theta}^{(2)}=\theta^{-1}(2f_2+stf_2')$$

$$\Delta L_2=O(\theta^{-2}), \quad \Delta P_2=O(\theta^{-1}), \quad \Delta Q_2=O(1)$$

$$R_2 = \theta(2Q_1Q_2L_1 + Q_1^2L_2) - P_{1\theta}Q_2 - P_{2\theta}^{(1)}Q_1 + Q_{1\theta}P_2 + Q_{2\theta}^{(1)}P_1$$

$$R_2 = O(\theta^{-3}\ln\theta), \quad \Delta R_2 = O(\theta^{-3}) \quad (s = \sin\omega, \ t = \cos\omega)$$

Putting $R_2 = 0$ and taking account of (2.7), we obtain the boundary-value problem for $f_2(\omega)$

$$(-4s^2 + 3s^4)f_2'' + st(4 - 9s^2)f_2' - 12s^2t^2f_2 = 0 \tag{2.12}$$

$$f_2(0) = f_2(\pi) = 0$$

The solution of problem (2.12) has the form $f_2(\omega) = k(2\sin^2\omega - \sin^4\omega)$, where k is an arbitrary constant.

Now, let $\psi_1 = \theta^{-1}\sin^3\omega$, $\psi_2 = k\ln\theta(2\sin^2\omega - \sin^4\omega)$, $\psi_3 = f_3(\omega)$, $\psi_0 = \psi_1 + \psi_2 + \psi_3$. Then

$$Y(\psi_0) = \theta^{-1}s^3 + k\ln\theta(2s^2 - s^4) + \alpha(3t - t^3 + 2) + f_3 + O(\theta\ln\theta)$$

$$L(\psi_0) = L_1 + L_2 + L_3 + \Delta L_3, \quad P(\psi_0) = P_1 + P_2 + P_3 + \Delta P_3$$

$$Q(\psi_0) = Q_1 + Q_2 + Q_3 + \Delta Q_3, \quad R(\psi_0) = R_3 + \Delta R_3$$

$$L_2 = 4\alpha^2k\theta^{-2}\ln\theta s^2t^2(1 - 4s^2), \quad P_2 = 16\alpha^2k\theta^{-1}\ln\theta s^5t^4$$

$$L_3 = \theta^{-2}[\alpha^2k(6s^2 - 15s^4 + 8s^6) + \alpha^{-1}a_1s^2t(2 - 15s^2 + 15s^4) - 3\alpha c_0s^4t +$$

$$+6\alpha s^4t(4 - 5s^2) + \alpha^2s^2f_3''], \quad Q_2 = k\ln\theta(8s^2 - 10s^4 + 4s^6)$$

$$P_3 = \theta^{-1}[18\alpha s^7t^3 + \alpha^{-1}a_1s^5t(4 - 12s^2 + 9s^4) + \alpha^2ks^5(8 - 16s^2 + 6s^4) +$$

$$+4\alpha^2s^4tf_3'], \quad Q_3 = 2\alpha(3t - t^3 + 2) + k(2s^2 - s^4) + 2f_3 + stf_3'$$

$$P_{2\theta}^{(1)} = 16\alpha^2k\theta^{-2}\ln\theta s^5t^4(4 - 9s^2), \quad Q_{2\theta}^{(1)} = 4k\theta^{-1}\ln\theta s^2t^4(4 - 6s^2)$$

$$P_{2\theta}^{(2)} = 16\alpha^2k\theta^{-2}s^5t^4, \quad Q_{2\theta}^{(2)} = k\theta^{-1}(8s^2 - 10s^4 + 4s^6)$$

$$P_{3\theta} = \theta^{-2}[36\alpha t^3(3s^7 - 5s^9) + \alpha^{-1}a_1t(16s^5 - 96s^7 + 168s^9 - 90s^{11}) +$$

$$+\alpha^2k(32s^5 - 136s^7 + 160s^9 - 54s^{11}) + 4\alpha^2s^5t^2f_3'' + 4\alpha^2s^4t(3 - 5s^2)f_3']$$

$$Q_{3\theta} = \theta^{-1}[-6\alpha s^4t + k(4s^2 - 8s^4 + 4s^6) + st(3 - 2s^2)f_3' + s^2t^2f_3'']$$

$$\Delta L_3 = O(\theta^{-1}\ln\theta), \quad \Delta P_3 = O(\ln^2\theta), \quad \Delta Q_3 = O(\theta\ln\theta)$$

$$R_3 = \theta(2Q_1Q_3L_1 + Q_1^2L_3) - P_{1\theta}Q_3 - (P_{2\theta}^{(2)} + P_{3\theta})Q_1 + Q_{1\theta}P_3 + (Q_{2\theta}^{(2)} + Q_{3\theta})P_1$$

$$R_3 = O(\theta^{-3}), \quad \Delta R_3 = O(\theta^{-2}\ln^2\theta)$$

Putting $R_3 = 0$ and taking account of (2.7), we obtain the boundary-value problem for $f_3(\omega)$

$$(1 + 3t^2)f_3'' + s^{-1}(5t - 9t^3)f_3' + 12t^2f_3 = \alpha^{-3}a_1F_1 + \alpha^{-1}c_0F_2 + \alpha^{-1}F_3 + \alpha F_4 + kF_5,$$

$$f_3(0) = f_3(\pi) = 0 \tag{2.13}$$

$$F_1 = 3s^2(-t + 8t^3 + 9t^5), \quad F_2 = 3s^2(t + 6t^3 + 9t^5), \quad F_3 = -2F_1$$

$$F_4 = 6t(1-4t-4t^2-t^4),\quad F_5 = -1-31t^2+t^4+15t^6$$

It can be shown that the functions

$$y_1(\omega) = 1-t^4,\quad y_2(\omega) = t+3t^3+\frac{3}{2}(1-t^4)\ln\frac{1+t}{1-t} \tag{2.14}$$

are solutions of the homogeneous equation which corresponds to (2.13) and $y_1(0) = y_1(\pi) = 0$, $y_2(0) = -y_2(\pi) = 4$. It can also be shown that one of the solutions of Eq. (2.13) has the form

$$f_3^0(\omega) = \alpha^{-3}a_1\varphi_1 + \alpha^{-1}c_0\varphi_2 + \alpha^{-1}\varphi_3 + \alpha\varphi_4 + k\varphi_5 \tag{2.15}$$

$$\varphi_1 = -\frac{1}{2}t - t^3 + \frac{3}{2}t^5,\quad \varphi_2 = -t - \frac{1}{2}t^3 + \frac{3}{2}t^5,\quad \varphi_3 = -2\varphi_1$$

$$\varphi_4 = -2 - \frac{3}{2}t - \frac{1}{2}t^3,\quad \varphi_5 = -\frac{8}{3} + (1-t^4)\ln s$$

and $f_3^0(0) = -4\alpha - 8/3k, f_3^0(\pi) = -8/3k$.

Imposing the condition $f_3^0(0) = -f_3^0(\pi)$ on the free parameter k, we obtain $k = -3/4\alpha, f_3^0 = -f_3^0(\pi) = -2\alpha$. It is not difficult to verify that the function $f_3(\omega) = f_3^0\omega) + 1/2\alpha y_2(\omega)$, defined by relations (2.14) and (2.15), is a solution of the boundary-value problem (2.13) when $k = -3/4\alpha$.

Hence

$$\psi_2 = -\frac{3}{4}\alpha\ln\theta(2\sin^2\omega - \sin^4\omega) \tag{2.16}$$

$$\psi_3 = \sin^2\omega\cos\omega\left[\alpha^{-3}a_1\left(-2+\frac{3}{2}\sin^2\omega\right)+\alpha^{-1}c_0\left(-\frac{5}{2}+\frac{3}{2}\sin^2\omega\right)+\right.$$
$$\left.+\alpha^{-1}(4-3\sin^2\omega)-\alpha\right]+\frac{3}{4}\alpha(1-\cos^4\omega)\ln\frac{1+\cos\omega}{(1-\cos\omega)\sin\omega} \tag{2.17}$$

The functions ψ_1, ψ_2, ψ_3 defined by (2.11), (2.16) and (2.17), satisfy conditions (2.7). By substituting $\psi = \psi_1 + \psi_2 + \psi_3$ into Eq. (2.1), the terms of the order of θ^{-4}, $\theta^{-3}\ln\theta$, θ^{-3} which occur when this is done are cancelled and a residual of the order of $\theta^{-2}\ln^2\theta$ remains. There are sufficient grounds to suppose that the process of finding the functions ψ_k by a successive reduction in the order of the residual in Eq. (2.1) can be continued and that the functions ψ_1, ψ_2, ψ_3 obtained are the first terms of the asymptotic expansion (2.5) (analysis shows that the function $\psi_1 = \theta^{-1}(\ln\theta)^m\sin^3\omega$ when $m \neq 0$ cannot serve as a basis for finding the successive terms of expansion (2.5)).

We shall assume that the function ψ^0 has been found to a sufficiently high accuracy. The function $\chi = \psi - \psi^0$ must serve as a solution of the boundary-value problem

$$L(\chi) = N(\psi^0+\chi) - L(\psi^0),\quad (\tau,\theta)\in\Sigma$$
$$N(\psi) = (P_\theta Q - PQ_\theta)/(Q^2\sin\theta);\quad \chi = -\psi^0 \quad \text{in } ABB_1CDA \tag{2.18}$$

Using the techniques described in [4], problem (2.18) can be reduced to solving an iterative sequence of linear difference equations. The accuracy of the resulting solution can be checked by comparing the values of $y(\tau,\theta)$ by integrating relations (2.2) along different trajectories. Having determined $\psi(\tau,\theta)$ and $r(\tau,\theta)$ using the method described and $x(\tau,\theta)$ by integrating the expression [11, 4]

$$x_\tau = [(M^2-1)\psi_\theta\cos\theta - \tau\psi_\tau\sin\theta + \mathrm{ctg}\,\theta P/Q](r\nu\tau^2)^{-1}$$
$$x_\theta = (\tau\psi_\tau\cos\theta - \psi_\theta\sin\theta)(r\nu\tau)^{-1} \tag{2.19}$$

there is no difficulty in finding all the necessary flow characteristics.

Suppose that v_a, ρ_a are the velocity and density of the fluid at infinity, p is the pressure, p_a and p_c are the values of p at infinity and on the free surface, R_0 is the radius of the base of the cone, R is radius of the cylinder to which the free surface is joined, C_x is the drag coefficient of the cone and σ is the pressure coefficient behind the cone

$$C_x = 4(\rho_a V_a^2 R_0^2)^{-1} \int_0^{R_0} (p - p_c) r dr, \quad \sigma = \frac{2(p_a - p_c)}{\rho_a V_a^2}$$

(The integral is evaluated along the generatrix BC.)

The drag of the solid of revolution formed by a semi-infinite cylinder and the leading part which is smoothly joined to it, around which there is a continuous axially symmetric flow of an ideal gas, is equal to zero. It follows from this that

$$C_x = \sigma R^2 / R_0^2 \tag{2.20}$$

It is convenient to use relation (2.20) as an additional check on the accuracy of the solution.

3. SOLUTION OF PROBLEM B FOR A PERFECT GAS WITH AN ADIABATIC INDEX $\gamma = 1.4$

The flow of an axially symmetric potential isentropic stream of an ideal perfect gas with an adiabatic index $\gamma = 1.4$ around a disc ($\theta_0 = \pi/2$) of radius R_0 was calculated using the Zhukovskii–Roshko scheme by the method which has been described, subject to the condition $M_c = 1$ for a series of values of $M_a = M_*$ in the range [0.5, 0.88] (M_a and M_c are the values of the Mach number at infinity and on the free surface). A mesh $I \times J = 50 \times 100$ was employed here in the domain of variation of the transformed variables of the velocity hodograph ξ, η (see [4]). The error with which condition (2.20) is satisfied in the case of the results obtained increases monotonically as M_a increases from 0.01% when $M_a = 0.5$ up to 0.8% when $M_a = 0.88$ (this sets an upper limit on the range of computational investigations).

The basic geometric characteristics of the leading parts of optimal semi-infinite solids of revolution are presented in Table 1. The values of parameter $\sigma = 2(p_a - p_c)/(\rho_a V_a^2)$, which characterizes the pressure on the lateral surface of the leading part of an optimal solid of revolution, can be calculated using the formula

$$\sigma = \frac{2}{\gamma M_a^2}\left[1 - \left(\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1} M_a^2\right)^{\gamma/(\gamma-1)}\right]$$

The contours of the leading parts of optimal bodies in the (x, r) meridian plane are shown in Fig. 2 for values of $M_* = 0.6$; 0.7; 0.75; 0.8; 0.83 (curves 1–5 respectively; $R = 1$).

In the case of an arbitrary solid of revolution, which is formed by a semi-infinite cylinder and a leading part which is smoothly joined to it, around which there is a continuous axially symmetric flow of a perfect gas with $\gamma = 1.4$, relations of the form

$$M_* \le G_1(R/L), \quad M_* \le G_2(R^2/S'), \quad M_* \le G_3(R^3/W')$$

Table 1

M_*	L/R_0	L/R	S'/R^2	W'/R^3
0.5	0.26353	0.22745	0.00895	0.05428
0.6	0.56128	0.43985	0.02692	0.15982
0.7	1.20394	0.82312	0.07202	0.41639
0.75	1.81351	1.13305	0.11591	0.65974
0.8	2.83888	1.58575	0.18753	1.04883
0.83	3.82819	1.96977	0.25305	1.39888
0.86	5.34163	2.49259	0.34694	1.89381
0.88	6.84913	2.96118	0.43418	2.34868

Table 2

M_*	a_1	a_2	a_3	a_4	a_5	a_6	a_7
0,5	-0,19488	0,05616	0,06095	-0,15976	0,18489	-0,11279	0,02852
0,6	-0,28986	0,02999	0,22089	-0,44649	0,48133	-0,28040	0,06822
0,7	-0,38834	-0,07723	0,59865	-1,09678	1,14069	-0,64622	0,15294
0,75	-0,43515	-0,17990	0,92127	-1,65434	1,71013	-0,96323	0,22600
0,8	-0,47634	-0,34008	1,40790	-2,50998	2,60348	-1,47173	0,34535
0,83	-0,49726	-0,47022	1,79171	-3,18016	3,29725	-1,86126	0,43449
0,86	-0,51402	-0,63263	2,25600	-3,97268	4,09445	-2,29189	0,52741
0,88	-0,52156	-0,77125	2,67094	-4,73264	4,92033	-2,77498	0,64150

hold, where G_1, G_2 and G_3 are monotonically decreasing functions of their arguments and a rigorous equality is only satisfied in the case of optimal bodies. The formulae

$$\begin{aligned}
G_1 &= (1-0.33587l_1+0.90137l_1^2-0.19790l_1^3)^{-1}, \quad l_1=(R/L)^{1/3} \\
G_2 &= (1-0.53319s_1+0.69406s_1^2-0.11091s_1^3)^{-1}, \quad s_1=(R^2/S')^{1/6} \\
G_3 &= (1-0.76759w_1+1.30968w_1^2-0.28216w_1^3)^{-1}, \quad w_1=(R^3/W')^{1/6}
\end{aligned} \tag{3.1}$$

are constructed using an approximation of the data from Table 1 and taking account of the fact that $G_1(0) = G_2(0) = G_3(0) = 1$.

The error of approximation of the calculated data in Table 1 by formulae (3.1) does not exceed 0.08% for G_1, 0.14% for G_2 and 0.16% for G_3.

In the neighbourhood of the points C and D (see Fig. 1a, b) $\psi \sim a(\tau-\tau_0)(\theta-\theta_0)$ and $\psi \sim b(\tau-\tau_0)\theta$, respectively ($a$ and b are certain constants). Hence, in accordance with (2.2) and (2.19) when $\theta_0 = \pi/2$ on the arc CD

$$\begin{aligned}
&x/R_0 = O(\mu^3), \quad r/R_0 - 1 = O(\mu^2) \quad \text{when } \mu \to 0 \\
&x/L - 1 = O(\theta^2), \quad r/L - R/L = O(\theta^3) \quad \text{when } \theta \to 0
\end{aligned} \tag{3.2}$$

where $\mu = \pi/2 - \theta$ ($x = 0, r = R_0$ at the point C and $x = L, r = R$ at the point D).

We now introduce the parametric variable β, assuming $x/L = p(a, \beta)/p(a, 0)$ in CD, where

$$p(a,\beta) = \frac{a^2\cos\beta}{2(1-2a^2\cos 2\beta + a^4)} - \frac{a}{8(1+a^2)}\ln\frac{1+2a\cos\beta+a^2}{1-2a\cos\beta+a^2}$$

and a is a fixed parameter in the range (0, 1) and β varies in the range $[0, \pi/2]$. It can be shown that $p(\alpha, \beta)$ is a monotonically decreasing function of β, $p(a, \pi/2) = 0$ and $p(a, \beta) = Q((\pi/2-\beta)^3$ as $\beta \to \pi/2$ and $p(a, \beta) - p(a, 0) = O(\beta^2)$ as $\beta \to 0$. By (3.2), $\theta = O(\beta)$ when $\beta \to 0$ and $\pi/2 - \theta = O(\pi/2 - \beta)$ when $B \to \pi/2$. The function $f(\beta)$ in the representation $r/R = f(\beta)$ must therefore satisfy the conditions $f(0) = 1, f'(\beta) = O(\beta^2)$ as $\beta \to 0$ and $f(\pi/2) = R_0/R, f(\beta) = O(\pi/2-\beta)$ as $B \to \pi/2$.

From the results of the calculations, taking account of what has been said above, approximate formulae were constructed for determining the shape of the leading parts of optimal semi-infinite solids of revolution in axially symmetric flow of a perfect gas when $\gamma = 1.4$ in the form

$$\frac{x}{R_0} = \frac{p(0,3;\beta)}{p(0,3;0)}\frac{L}{R_0}, \quad \frac{r}{R_0} = \frac{R}{R_0}\left(1+\sum_{k=1}^{7} a_k \sin^{2k+1}\beta\right), \quad \beta \in [0, \pi/2] \tag{3.3}$$

The values of a_k are shown in Table 2 (L/R_0 and L/R are shown in Table 1).

Suppose the ε_1 is the maximum relative error in the approximation of the quantity r by formulae (3.3) with respect to β (for fixed M_*) and that ε_2 is the maximum error with respect to β in the approximation of the quantity θ: $\varepsilon_2 = |\theta - \operatorname{arctg}(r_\beta/x_\beta)|$. As M_* increases, running through the values indicated in Tables 1 and 2, the magnitudes of ε_1 and ε_2 increase monotonically: the first from 10^{-5} to 7×10^{-5} and the second from 2×10^{-4} to 15×10^{-4}. The use of formulae (3.3) in the case of a spline approximation of the data from Tables 1 and 2 for intermediate values of M_* usually leads to an increase in the values of ε_1 and ε_2, but ε_1 does not increase by more than an order of magnitude while ε_2 increases by factor of no more than 2. This enables one to construct optimal semi-infinite solids of revolution in an axially symmetric flow of a perfect gas when $\gamma = 1.4$ for arbitrary values of M_* lying in the range [0.5; 0.88].

Computational investigations were carried out in [4–9] to determine the critical Mach number for certain shapes of the leading part of a semi-infinite solid of revolution in a flow of perfect gas with $\gamma = 1.4$. A half (up to the

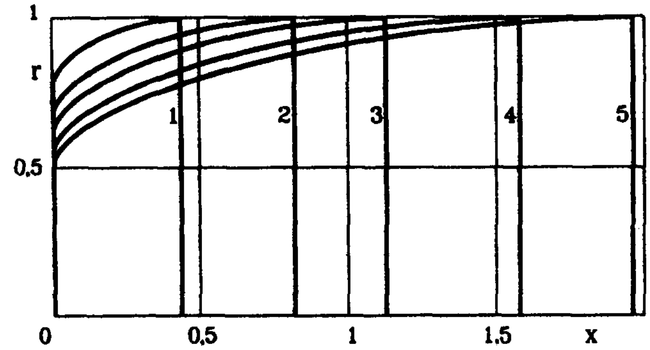

Fig. 2.

centre section) of the solid of revolution obtained in [12] as a result of the solution of a problem on the cavitational flow of an axially symmetric stream of an incompressible fluid around a disc or sphere, using the Ryabushinskii scheme, was considered as the leading part of the body (using the first initials of the authors of [12] and [7–9], for brevity, we shall call such bodies *KVK* bodies with plane and spherical bluntness). "Parabolic" bodies with a generatrix equation $r/R = [(x/L)(2-x/L)]^n$, where n is a positive constant, when $0 \leqslant x \leqslant L$ and $r/R = 1$ when $x > L$ were also considered. It was established that *KVK* bodies with a plane bluntness possess the greatest critical Mach number for a wide range of variation in the parameter L/R among the listed shapes of bodies.

Thus [9], when $L/(2R) = 0.8672 M_* = 0.786$ for a *KVK* body with plane bluntness, $M_* = 0.759$ for a *KVK* body with spherical bluntness, $M_* = 0.752$ for a parabolic body when $n = 0.3$ and $M_* = 0.742$ for a parabolic body when $n = 0.5$. Computational and experimental investigations have also been carried out to determine the drag of the above-mentioned solids of revolution when $L/(2R) = 0.8672$ in an air flow and it was established that *KVK* bodies with plane bluntness have minimum drag over the range of free-stream Mach numbers $M_* < M_a < 0.97$.

It is obvious that bodies, for which the shape of the leading section is obtained from the solution of the problem of the flow of an incompressible fluid around a disc using the Ryabushinskii scheme, are not absolutely optimal with respect to M_* and differ from bodies whose construction is dealt with in this paper. For instance, in the case of a body with a length of the leading section $L/(2R) = 0.8672$, the method developed in this paper gives max M_* $= 0.8127$. The dimensionless coordinates x/R and r_1/R of the contour of the leading section of a *KVK* body with plane bluntness when $L/(2R) = 0.8672$, borrowed from [9], are presented below.

x/r	0	0.0710	0.2656	0.3881	0.5593	0.7995	1.1350	1.6054	1.7344
r_1/R	0.5420	0.6396	0.7572	0.8070	0.8607	0.9165	0.9669	0.9984	1
$\Delta r/R$	0.0016	0.0075	0.0145	0.0168	0.0176	0.0170	0.0122	0.0020	0

The values of the coordinates r/R of the contour of the body which is optimal with respect to M_* when $L/(2R)$ $= 0.8672$ are obtained by subtracting the corresponding values of $\Delta r/R$ from r_1/R. The body which is optimal with respect to M_* lies completely within a *KVK* body with plane bluntness.

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